Collapse Dynamics of a Star of Dark Matter and Dark Energy

Subenoy Chakraborty* **and** **Tanwi Bandyopadhyay**
Department of Mathematics, Jadavpur University, Calcutta-32, India.
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In this work, we study the collapse dynamics of an inhomogeneous spherically symmetric star made of dark matter (DM) and dark energy (DE). The dark matter is taken in the form of a dust cloud while anisotropic fluid is chosen as the candidate for dark energy. It is investigated how dark energy modifies the collapsing process and is examined whether dark energy has any effect on the Cosmic Censorship Conjecture. The collapsing star is assumed to be of finite radius and the space time is divided into three distinct regions Σ and $V^{\pm}$, where Σ represents the boundary of the star and $V^{-}(V^{+})$ denotes the interior (exterior) of the star. The junction conditions for matching $V^{\pm}$ over Σ are specified. Role of Dark energy in the formation of apparent horizon is studied and central singularity is analyzed.

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I. INTRODUCTION

The collapse dynamics in the framework of Einstein gravity was started long ago by Oppenheimer and Snyder [1]. An extensive study of gravitational collapse for matter cloud in the form of pressure-less dust has been done in recent years [2-6] (for recent reviews see [7]). These studies show the validity of Cosmic Censorship Conjecture in six and higher dimensions under reasonable physical conditions, the role of initial data on the end state of collapse and some dynamical symmetries [8] on the initial data set. Also the collapse dynamics has considerable astrophysical significance.

So far, these studies are mostly concentrated to dust cloud and matter with pressure components [9-13]. Also there are few works [14-15] where in addition to dust matter, one has cosmological constant with it. Recently, Madhav, Goswami and Joshi [16] have studied in details the effect of a cosmological constant in the background of asymptotically anti de-Sitter (or de-Sitter) space -time. They have treated the cosmological constant as a dark energy component, motivated by the recent astronomical observations [17] of high redshift type Ia Supernova. Usually, the dark energy behaves as the source of repulsive gravity and is considered to be important in the present era to justify the undergoing accelerated expansion of the universe.

Thus, it is interesting to study gravitational collapse of a matter cloud containing dark energy. In the present work, we examine the collapsing process of an inhomogeneous spherically symmetric star whose matter inside contains a combination of dark matter (dust) and dark energy. We have studied in two different sections the collapsing scenario for TBL model and general spherically symmetric model. The formation of apparent horizon has been investigated in details for various cases and it is examined whether bouncing solutions are possible or not. The paper ends with a short discussion and concluding remarks.

II. BASIC EQUATIONS

The metric ansatz of a general spherically symmetric metric can be written as

$$ds^2 = -e^{2\gamma}dt^2 + e^{2\alpha}dr^2 + R^2 d\Omega^2 \tag{1}$$

where α, γ and R are functions of t and r and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the usual line element on a unit 2-sphere. The form of the energy momentum tensor is taken as

* subenoy@iucaa.ernet.in

$$T^{\nu}_{\mu\,(total)} = T^{\nu}_{\mu\,(DM)} + T^{\nu}_{\mu\,(DE)} \tag{2}$$

Here the coordinates (t, r, θ, ϕ) are taken to be comoving and the components of the above energy momentum tensor in this comoving frame are

$$T^{\nu}_{\mu\,(DM)} = diag(-\rho_{_{DM}}, 0, 0, 0) \tag{3}$$

and

$$T^{\nu}_{\mu\,(DE)} = diag(-\rho, p_r, p_t, p_t) \tag{4}$$

The quantities $\rho_{_{DM}}, \rho, p_r$ and p_t are respectively the dark matter density, dark energy matter density, radial and tangential pressure for dark energy and are functions of r, t. In order to satisfy the weak energy condition for the whole matter we have

$$T^{\nu}_{\mu\,(total)} V^{\mu} V^{\nu} \geq 0 \tag{5}$$

for any time-like vector V^{μ}. In explicit form this means

$$\rho_{_T} \geq 0, \quad \rho_{_T} + p_r \geq 0, \quad \rho_{_T} + p_t \geq 0 \tag{6}$$

where $\rho_{_T} = \rho_{_{DM}} + \rho$, is the total matter density. Now the explicit form of the Einstein's field equations for the metric (1) with matter field given by (2) are (choosing $8\pi G = c = 1$)

$$e^{-2\gamma}\left(\frac{\dot{R}^2}{R^2} + 2\dot{\alpha}\frac{\dot{R}}{R}\right) + \frac{1}{R^2} - e^{-2\alpha}\left(2\frac{R''}{R} + \frac{R'^2}{R^2} - 2\alpha'\frac{R'}{R}\right) = \rho_{_T} \tag{7}$$

$$-e^{-2\gamma}\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - 2\dot{\gamma}\frac{\dot{R}}{R}\right) - \frac{1}{R^2} + e^{-2\alpha}\left(\frac{R'^2}{R^2} + 2\gamma'\frac{R'}{R}\right) = p_r \tag{8}$$

$$-e^{-2\gamma}\left(\ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha}\dot{\gamma} + \frac{\ddot{R}}{R} + \dot{\alpha}\frac{\dot{R}}{R} - \dot{\gamma}\frac{\dot{R}}{R}\right) + e^{-2\alpha}\left(\gamma'' + {\gamma'}^2 - \alpha'\gamma' + \frac{R''}{R} + \gamma'\frac{R'}{R} - \alpha'\frac{R'}{R}\right) = p_t \tag{9}$$

and

$$\dot{R}' - \dot{\alpha}R' - \gamma'\dot{R} = 0 \tag{10}$$

where over dot(.) and dash(') stand for partial derivatives with respect to t and r respectively. We now introduce the mass function $m(t, r)$, defined by

$$m(t, r) = R(1 + e^{-2\gamma}\dot{R}^2 - e^{-2\alpha}{R'}^2) \tag{11}$$

which can be interpreted as the total mass inside the comoving radius r at the instant t. This definition was first introduced by Cahill and McVittie [18] and since then it has been widely used in different context. Also, for asymptotically flat space times, it corresponds to the correct Bondi mass at infinity. If the boundary of the collapsing cloud is leveled by the comoving coordinate r_Σ, then the total mass of the collapsing star at the instant t is given by

$$m_\Sigma(t) = m(t, r_\Sigma) \tag{12}$$

For regularity on the initial hypersurface $t = t_i$, we must choose $m(t_i, 0) = 0$.

Using this mass term, the field equations (7) and (8) can be written in compact form as

$$\rho_{_T} = \frac{m'}{R^2R'}, \qquad\qquad p_r = -\frac{\dot{m}}{R^2\dot{R}} \tag{13}$$

As the matter field is a mixture of dark matter (inhomogeneous dust) and dark energy (anisotropic fluid), so from energy momentum conservation relation ${T_\mu}^\nu{}_{;\nu} = 0$, we obtain

$$\dot{\rho_{_T}} + \rho_{_T}\left(\dot{\alpha} + 2\frac{\dot{R}}{R}\right) + \dot{\alpha}p_r + 2p_t\frac{\dot{R}}{R} = 0 \tag{14}$$

and

$$p_r' + (\rho_{_T} + p_r)\gamma' - 2(p_t - p_r)\frac{R'}{R} = 0 \tag{15}$$

If $Q(t,r)$ denotes the interaction between the dust cloud and dark energy, then a self consistent conservation equation for each of them can be separately written as (see eqn(14))

$$\dot{\rho_{_{DM}}} + \left(\dot{\alpha} + 2\frac{\dot{R}}{R}\right)\rho_{_{DM}} = Q(t,r) \tag{16}$$

and

$$\dot{\rho} + \left(\dot{\alpha} + 2\frac{\dot{R}}{R}\right)\rho + \dot{\alpha}p_r + 2\frac{\dot{R}}{R}p_t = -Q(t,r) \tag{17}$$

Thus we have six independent differential equations namely equations (10), (11), (13), (16) and (17) containing eight unknowns (geometrical or physical) namely, $\rho_{_{DM}}, \rho, p_r, p_t, Q(t,r), \alpha, \gamma$ and R, giving us freedom to choose two free functions. In particular, for a given initial data satisfying the weak energy condition, the choice of these free functions will determine the matter distribution and metric of the space time and hence a particular dynamical evolution of the initial data.

On the initial hypersurface: $t = t_i$ we have to specify eight functions of r namely $\rho_{_{DM_0}}(r), \rho_0(r), p_{r_0}(r), p_{t_0}(r), \gamma_0(r), \alpha_0(r), R_0(r)$ and $Q_0(r)$. By proper scaling of the radial coordinate r, we can choose $R_0(r) = r$, without any loss of generality. It is to be noted that the remaining seven (initial data) functions are not independent but are related by the field equations. Further, to preserve regularity and smoothness of the initial data, we have to assume that the initial pressures at the regular centre $r = 0$ should satisfy

i) gradient of the initial pressures vanish at the centre i.e, $p_{r_0}'(0) = p_{t_0}'(0) = 0$

ii) the initial pressure should be isotropic at the centre i.e, $p_{r_0}(0) = p_{t_0}(0)$.

From the field equations (13), we see that the total matter density becomes infinity either when $R = 0$ or $R' = 0$ or both. The case for $R' = 0$ is termed as shell crossing singularity which is not of much physical interest as it is gravitationally weak in nature and is removable singularity. The other one (which we shall discuss here) is termed as shell focusing singularity where the physical radius of all matter shells goes to zero ($R = 0$). So we shall assume $R' > 0$ at all instant to avoid any crossing of the shells. If $t_s(r)$ denotes the instant when a shell at coordinate radius r collapse to $r = 0$, then we have $R(t_s(r), r) = 0$. (Note that the singularity time t_s is a function of r due to the inhomogeneity of the space-time). In the subsequent sections we shall consider the collapse dynamics for two separate assumptions namely

i) a geometrical assumption: $\gamma = 0$ (LTB model) and

ii) a physical assumption: $p_r = 0$.

III. COLLAPSING PROCESS IN LTB MODEL WITH DM AND DE

The metric of the space time describing the collapsing matter in the standard Lemaitre-Tolman-Bondi (LTB) model is given by

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2 d\Omega^2 \tag{18}$$

The simplified field equations are

$$\frac{\dot{R}^2}{R^2} + 2\frac{\dot{R}}{R}\frac{\dot{R}'}{R'} - \frac{f}{R^2} - \frac{f'}{RR'} = \rho_T \tag{19}$$

$$\frac{f}{R^2} - \frac{\dot{R}^2}{R^2} - 2\frac{\ddot{R}}{R} = p_r \tag{20}$$

and

$$\frac{f'}{2RR'} - \frac{\dot{R}}{R}\frac{\dot{R}'}{R'} - \frac{\ddot{R}}{R} - \frac{\ddot{R}'}{R'} = p_t \tag{21}$$

with energy conservation equations

$$\dot{\rho_{DM}} + \left(2\frac{\dot{R}}{R} + \frac{\dot{R}'}{R'}\right)\rho_{DM} = Q(t,r) \tag{22}$$

$$\dot{\rho} + \left(2\frac{\dot{R}}{R} + \frac{\dot{R}'}{R'}\right)\rho + p_r\frac{\dot{R}'}{R'} + 2p_t\frac{\dot{R}}{R} = -Q(t,r) \tag{23}$$

and

$$p_r' = 2(p_t - p_r)\frac{R'}{R} \tag{24}$$

From equation (24), it is to be noted that

a) anisotropy of pressure implies the inhomogeneity of the pressure and vice-versa
b) dark energy model with vanishing radial pressure and non-zero tangential pressure ($p_r = 0, p_t \neq 0$) is not possible.

From the recent study of the role of pressure in quasi-spherical gravitational collapse, we choose the radial pressure as [12]

$$p_r(t,r) = -\frac{g(r)}{R^n} \tag{25}$$

Then from the conservation equation (24), the tangential pressure has the expression

$$p_t(t,r) = -\frac{g(r)}{R^n}\left(1-\frac{n}{2}\right) - \frac{g'(r)}{2R^{n-1}R'} \tag{26}$$

(Note that we have taken $-$ve sign for p_r as pressure is $-$ve for dark energy). The arbitrary function $g(r)$ should be $\bigcirc(r^{n+2})$ near $r=0$ for the regularity and smoothness of the initial pressure. Solving the field equations (13) we have the expression for mass function and total matter density as

$$m(t,r) = m_0(r) + \frac{g(r)}{(3-n)R^{n-3}}, \qquad (n \neq 3) \tag{27}$$

and

$$\rho_T(t,r) = \frac{m_0'(r)}{R^2R'} + \frac{g'(r)}{(3-n)R^{n-1}R'} + \frac{g(r)}{R^n} \tag{28}$$

where $m_0(r)$ is an arbitrary function of r, restricted by the energy conditions. The evolution equation for the area radius R can be obtained from the field equation (20) (or from the definition of the mass function (11)) as

$$\dot{R}^2 = f(r) + \frac{m_0(r)}{R} + \frac{g(r)}{(3-n)R^{n-2}} \tag{29}$$

Now using the scaling independence of the coordinate r [16] let us write

$$R(t,r) = rv(t,r) \tag{30}$$

so that

$$v(t_i,r) = 1, \qquad v(t_s,r) = 0, \qquad \dot{v} < 0 \tag{31}$$

Then from (29) the time evolution equation for v is

$$\dot{v}^2 = \left[\frac{f(r)}{r^2}v + \frac{m_0(r)}{r^3} + \frac{g(r)}{(3-n)r^{n-1}}v^{3-n}\right]/v = \frac{V(r,v)}{v} \tag{32}$$

This is the evolution equation of a particular shell and $V(r,v)$ may be termed as an effective potential.

One of the main questions that we shall address here is how the collapse dynamics gets modified due to the presence of dark energy. In case of pure dust collapse, once the collapse initiates, starting from an initial regular hypersurface, the cloud necessarily collapses to a singularity due to gravitational attraction without any reversal or bounce. On the other hand, the negative pressure of the dark energy produces repulsive gravitational force and slows down the collapsing process. So it is expected that, due to the presence of dark energy, the collapsing process may not be smooth. Thus, it is interesting to study in details the qualitative nature of the effective potential $V(r,v)$ for which the allowed regions of motion correspond to $V(r,v) \geq 0$. In fact initially, when the collapse starts, we have $\dot{v} < 0$ and we will have a rebounce if we get $\dot{v} = 0$ before the shell has become singular. In other words, study of various evolution for a particular shell is equivalent to study the nature of the roots of the equation $V(r,v) = 0$ for fixed r.

To get a clear idea, let us consider a smooth initial data for which initial density, radial pressure and curvature are smooth near $r=0$ and the corresponding functions will have power series expansion as follows:

$$\left.\begin{aligned}
\rho_{T0}(t_i,r) &= \rho_{00}+\rho_{02}r^2+\rho_{04}r^4+\ldots\ldots.. \\
p_{r0}(t_i,r) &= -p_{02}{}^{(r)}r^2-p_{04}{}^{(r)}r^4-p_{06}{}^{(r)}r^6-\ldots\ldots... \\
f(r) &= f_2r^2+f_4r^4+\ldots\ldots..
\end{aligned}\right\} \tag{33}$$

As we are concentrating on the evolution of the shells near $r=0$ so we may neglect here higher order terms in the above expansions. Thus near the centre ($r << r_\Sigma$) equation (32) becomes

$$\dot{v}^2 = \left[f_0 v + \frac{\rho_{00}}{3} + \frac{p_{02}{}^{(r)}r^3}{(3-n)}v^{3-n}\right]/v \tag{34}$$

The numerator of the right hand side of the above equation (i.e the effective potential $V(r,v)$) is a polynomial in v and in general may or may not have positive real roots which correspond to physical cases. As $V(r,0)=\frac{\rho_{00}}{3}>0$ so any region between $R=rv=0$ and the first positive zero of $V(r,v)$ always becomes singular during collapse while if there are two consecutive positive real roots then the region between them is forbidden as $\dot{v}^2<0$ there. Hence a particular shell will bounce if it lies in the initial epoch ($v=1$) to the right of the second positive root. We shall present various possibilities for $n=1$ and derive the necessary conditions below:

a) $f_0=0$**: (Marginally Bound Case)**

Here as both ρ_{00} and $p_{02}{}^{(r)}$ are positive so both roots of the quadratic equation are complex and hence $\dot{v}^2$ is always positive and no bounce occurs. Hence a singularity always forms from the initial collapse.

b) $f_0>0$:

Both roots will be real and negative if ${f_0}^2 \geq \frac{2}{3}\rho_{00}\ p_{02}{}^{(r)}r^2$, otherwise they are complex in nature.

c) $f_0<0$:

For the restriction ${f_0}^2 \geq \frac{2}{3}\rho_{00}\ p_{02}{}^{(r)}r^2$, both roots are real and positive, otherwise they are complex. So bounce will be possible in this case.

Horizons:

For observers at infinity, the event horizon plays an important role in characterizing the nature of the singularity. As the calculations are complex in nature, so instead of event horizon, one considers a trapped surface which is a compact space-time 2-surface with normals on both sides as future pointing converging null geodesic families. In fact, if ($r=$constant, $t=$constant) the 2-surface $S_{r,t}$ is a trapped surface then it and its entire future development lie behind the event horizon provided the density falls off fast enough at infinity. As apparent horizons (AH) are the boundaries of trapped regions (surfaces), so in the present case it can be written as [19]

$$\text{AH}: \qquad g^{\mu\nu}R_{,\mu}R_{,\nu}=0$$

Using the metric form (1) one gets

$$-e^{-2\gamma}\left(\frac{\partial R}{\partial t}\right)^2+e^{-2\alpha}\left(\frac{\partial R}{\partial r}\right)^2=0,$$

which with the help of the mass function simplifies to

$$1-\frac{m(t,r)}{R}=0 \tag{35}$$

In the present case (i.e, for LTB model) the apparent horizon is characterized by

$$1-\frac{m_0(r)}{R}-\frac{g(r)}{3-n}\frac{1}{R^{n-2}}=0 \qquad (n\neq 3) \tag{36}$$

We shall now discuss the formation of apparent horizon for different values of n:

$n=1$:

In this case, equation (36) is a quadratic equation in R, having real positive roots if

(i) $0<m_0 g<\frac{1}{2}$ and (ii) $g(r)>0$

and these horizons are given by

$$\left.\begin{array}{l} R_c(r)=\left\{1-\sqrt{1-2m_0 g}\right\}/g \\ R_b(r)=\left\{1+\sqrt{1-2m_0 g}\right\}/g \end{array}\right\} \tag{37}$$

These are usually termed as the cosmological and the black hole horizons [19].

$n=2$:

Equation (36) reduces to a linear equation in R, so there is only one horizon namely

$R_{bc}(r)=\frac{m_0(r)}{1-g(r)}$ (assuming $g(r)<1$).

$n=4$:

Again equation (36) is a quadratic equation in R, having two horizons if $|m_0|>2\sqrt{g}$.

$n=5$:

Here equation (36) simplifies to a cubic equation in R as

$$2R^3-2m_0(r)R^2+g(r)=0 \tag{38}$$

In the following we shall discuss the nature of the roots of the cubic equation for various possibilities:

i) For $g(r)<\frac{4{m_0}^3}{27}\left(1+\frac{1}{3\sqrt{3}}\right)$,

there are two positive roots of (38), which correspond to two apparent horizons namely the cosmological horizon and the black hole horizon given by

$$\left.\begin{array}{l} R_c(r)=\frac{m_0(r)}{3}+\frac{2{m_0}^3}{(27)^{3/2}}Cos\left[\frac{1}{3}Cos^{-1}\left\{-\frac{(27)^{3/2}}{2{m_0}^3}\left(\frac{g}{2}-\frac{2{m_0}^3}{27}\right)\right\}\right] \\ R_b(r)=\frac{m_0(r)}{3}+\frac{2{m_0}^3}{(27)^{3/2}}Cos\left[\frac{4\pi}{3}+\frac{1}{3}Cos^{-1}\left\{-\frac{(27)^{3/2}}{2{m_0}^3}\left(\frac{g}{2}-\frac{2{m_0}^3}{27}\right)\right\}\right] \end{array}\right\} \tag{39}$$

ii) When $g(r)=\frac{4{m_0}^3}{27}\left(1+\frac{1}{3\sqrt{3}}\right)$,

then there is only one positive root (corresponds to a single apparent horizon) of the above cubic equation given by

$$R_{bc}(r)=\frac{m_0}{3}+{m_0}^3/(27)^{\frac{3}{2}} \tag{40}$$

iii) If $g(r) > \frac{4{m_0}^3}{27}\left(1+\frac{1}{3\sqrt{3}}\right)$,

then there are no possible roots and hence there are no apparent horizons.

Further, if $t_{ah}(r)$ be the time of formation of apparent horizon then from the evolution equation (34) we have

$$t_{ah}(r) - t_s(r) = -\int_0^{v_{ah}} \frac{\sqrt{v}dv}{\left[f_0 v + \frac{\rho_{00}}{3} + \frac{{p_{02}}^{(r)} r^3}{3-n} v^{3-n}\right]^{\frac{1}{2}}} \tag{41}$$

where we have considered shells close to $r = 0$. Here $t_s(r)$ is the time of formation of singularity of a shell at comoving radius r and $v_{ah} = R_b$ (or R_c)$/r$.

It is to be noted that the presence of the dark energy in the form of anisotropic fluid characterizes the formation of apparent horizon and also influences the time difference between the formation of apparent horizon and singularity formation.

The above time difference will characterize the final end state of gravitational collapse and also the nature of the resulting singularity. If the formation of the horizon precedes the formation of the central singularity, then the singularity will necessarily be covered, i.e, black hole will be formed. On the other hand, if the time difference is reversed then end state of collapse leads to a naked singularity. As this characterization of the singularity is local, so for global visibility, we should examine whether it is possible to have future directed null geodesics that terminate at the singularity in the past. Since analysis of null geodesic has been done widely so we are not presenting it here.

IV. GENERAL SPHERICALLY SYMMETRIC COLLAPSING MODEL WITH $p_r = 0, p_t \neq 0$ FOR DARK ENERGY

Over the last few years spherically symmetric collapse with anisotropic pressure in the form of vanishing radial pressure have been studied in details. The main objective of the present study is to examine whether bouncing situation is possible due to the presence of dark energy and to investigate singularity formation conditions. Also we try to understand how dark energy affects the junction conditions at the boundary of the collapsing star to an exterior region, the formation of trapped surfaces (apparent horizon) and the nature of the central shell focusing singularity.

For general spherically symmetric model (given by equation (1)), if we assume the radial pressure to be zero then from the field equations (13) we have

$$\rho_{_T}(r,t) = \frac{m'(r)}{R^2 R'} \tag{42}$$

and from the conservation equation (15) using (42) we get

$$p_t = \frac{m'(r)\gamma'}{2RR'^2} \tag{43}$$

Here $m(r)$ is an arbitrary function of r, satisfying the energy conditions. For the remaining freedom to choose one function, we take the metric coefficient $\gamma(r,t)$ in the specific form

$$\gamma(r,t) = \gamma_0(t) + \beta(R) \tag{44}$$

(motivation for choosing such a form is given in ref. [16]).

Using this choice of γ in equation (10) we have the other metric coefficient as

$$e^{\alpha} = \frac{R'(t,r)}{b(r)e^{\beta(R)}} \tag{45}$$

with $b(r)$ an arbitrary C^2 function of r.

These choices of the metric coefficients when substituted in the definition of the mass function (i.e, equation (11)), we obtain the evolution equation for the area radius R as

$$\dot{R}^2 = \frac{e^{2\beta(R)}}{R}\left[m(r) + \left\{b^2(r)e^{2\beta(R)} - 1\right\}R\right] \tag{46}$$

where a multiplicative function of time has been eliminated by a suitable scaling of the time coordinate. As in the previous section, introducing $R(t,r) = rv(t,r)$ the above equation transforms to

$$\dot{v}^2 = \frac{e^{2\beta(rv)}}{v}\left[\frac{m(r)}{r^3} + \frac{\left\{b^2(r)e^{2\beta(rv)} - 1\right\}}{r^2}v\right] \tag{47}$$

To study this evolution equation near the singularity ($r = 0$), one should consider smooth initial data (i.e, initial density and pressure) with power series expansion (near $r = 0$)

$$\left.\begin{aligned} &\rho_i = \rho_{_T}(t_i, r) = \rho_0 + \rho_2 r^2 + \rho_4 r^4 + \ldots.. \\ &p_{ti} = p_t(t_i, r) = -p_{t2}r^2 - p_{t4}r^4 - p_{t6}r^6 - \ldots.. \qquad \text{and} \\ &b^2(r) = 1 + b_{02}(r)r^2 + b_{04}(r)r^4 + \ldots.. \end{aligned}\right\} \tag{48}$$

where the first term in the series expansion for $b^2(r)$ is chosen to be unity as the metric looks Minkowskian near the centre $r = 0$. Thus close to $r = 0$, neglecting higher order terms in the above expression (47) for $\dot{v}$, one gets

$$\dot{v}^2 = \left(1 - \frac{2p_{t2}}{\rho_0}r^2v^2\right)\left(\frac{\rho_0}{3} + b_{02}v - \frac{2p_{t2}b(r)}{\rho_0}v^3\right)/v \tag{49}$$

During the collapsing process the scale factor v decreases from unity (at the initial epoch) to zero (at the time of singularity), so the first factor in the numerator of the right hand side of (49) is initially positive (as it represents $e^{2\gamma}$). Hence this factor is positive definite during the collapse dynamics. Thus, whether a shell bounces or not is completely determined by the second factor in the effective potential.

The second factor, which is a cubic equation in v, has in general three distinct roots. As positive real roots are only physically admissible, so for different situations we examine below whether positive roots are possible or not. As the term in quadratic power of v (i.e, v^2) is absent, so whenever all the three roots are real, there should be at least one positive and one negative root.

TABLE (Analysis of the roots of the equation)

Condition	**Nature of the roots**	**Physical Consequences**
(i) $b_{02}(r) \geq 0$	Exactly one possible root (say $\alpha(r)$) and the two other roots are either negative or complex conjugate	The physically allowed range of v is $[0, \alpha]$. If $\alpha \geq 1$, then there will always be a singularity while $\alpha < 1$ implies an unphysical situation initially as $\dot{v}^2 < 0$, i.e, all shells in the allowed dynamical space $[0, \alpha]$ becomes singular starting from initial collapse
(ii) $b_{02}(r) \leq 0$ (a) $\rho_0 > \frac{2}{3}\frac{{b_{02}}^3(r)}{p_{t2}b(r)}$	No positive real root	Singularity is always the final outcome of the collapse
(b) $\rho_0 < \frac{2}{3}\frac{{b_{02}}^3(r)}{p_{t2}b(r)}$	Two positive roots $v_1(r)$ and $v_2(r)$ $(v_1 < v_2)$ where $v_1 = \lambda\ Cos\psi$, $v_2 = \lambda\ Cos(\psi + \frac{4\pi}{3})$ with $\lambda = \sqrt{\frac{2}{3}\frac{b_{02}}{p_{t2}b(r)}}$ and $3\psi = Cos^{-1}\left[-\sqrt{\frac{3\rho_0 p_{t2}b(r)}{2{b_{02}}^3(r)}}\right]$	Allowed space for dynamical evolution is when v lies in the range $[0, v_1]$ and $[v_2, \infty)$. The value of v in (v_1, v_2) is not allowed as $\dot{v}^2 < 0$. Initially, shells for $v \in [0, v_1]$ always become singular but shells for $v \in [v_2, \infty)$ will undergo a bounce and subsequent expansion starts from initial collapse
(c) $\rho_0 = \frac{2}{3}\frac{{b_{02}}^3(r)}{p_{t2}b(r)}$	Two equal positive roots	There is no forbidden region and bounce will occur if $\lambda < 2$

In case (b)(of (ii)), there are two positive real roots and bounce will occur when the area radius approaches $R_b = rv_2$. As in this case both singularity and bounce may occur so the explicit conditions for a particular shell to become singular or undergo a bounce can be stated as

i) For singularity: $\lambda\ Cos\psi > 1$

ii)For bounce: $\lambda\ Cos(\psi + \frac{4\pi}{3}) < 1$

So far, we have analyzed the evolution of shells close to the centre. But it is difficult to study (analytically) shells far from the centre and simple expression for the singularity or bounce is not possible, only one has to use numerical methods. However, the above analysis can be applicable to the entire cloud within the boundary

$r = r_\Sigma$, provided either the initial area radius r_Σ is itself small compared to the initial data coefficients, i.e, $\rho_i {r_\Sigma}^i$, $p_{ti} {r_\Sigma}^i << 1$ or the initial data are such that higher order coefficients in the power series expansion are identically zero i.e, $\rho_i = 0 = p_{ti}, \forall \;\; i \geq 4$ and $\rho_2 {r_\Sigma}^2$, $p_{t2} {r_\Sigma}^2 << 1$. Moreover, if the whole collapsing cloud undergoes bounce starting from the initial collapse (i.e, case(b) of (ii) holds for all r in $0 \leq r < r_\Sigma$) then the sufficient condition for avoiding shell crossing is $\frac{v_2(r+\delta)}{v_2(r)} \geq 1, \forall \;\; r \; \in \; [\; 0, \; r_\Sigma)$ and $\delta > 0$, a small increment.

V. SPACE-TIME MATCHING

We shall now show the space-time matching of the collapsing star with the exterior vacuum. First of all, we shall find the matching conditions for a general exterior space-time and then for Schwarzschild space-time as a particular case. For smooth matching we shall use the Israel-Darmois [20] junction conditions on the boundary $\sum$.

Let the interior of the star is denoted by V^- and V^+ is the exterior vacuum space-time region with $\sum$, the surface of the star as the common boundary. So the interior space-time (V^-) is described by the metric given in equation (1) while that of the exterior vacuum (V^+) is taken as

$$V^+ : \;\; {ds_+}^2 = -A^2(T,\Lambda)\; dT^2 + B^2(T,\Lambda)\; [d\Lambda^2 + \Lambda^2 d\Omega^2] \tag{50}$$

The equation of the bounding surface $\sum$, considering it as an embedding in the interior or exterior space-times are given by

$$\Sigma^- : \; r = r_\Sigma; \qquad \Sigma^+ : \; \Lambda = \Lambda_\Sigma(T) \tag{51}$$

with r_Σ, a constant. Then the metric in $\sum$ relative to the coordinates of V^- and V^+ are respectively

$$\Sigma^- : \; {ds_{\Sigma^-}}^2 = e^{-2\gamma} dt^2 + R^2(t, r_\Sigma)\; d\Omega^2 \tag{52}$$

and

$$\Sigma^+ : \; {ds_{\Sigma^+}}^2 = -A^2(T,\Lambda_\Sigma)\; dT^2 + B^2(T,\Lambda_\Sigma)\; [{d\Lambda_\Sigma}^2 + {\Lambda_\Sigma}^2\; d\Omega^2] \tag{53}$$

According to Israel-Darmois, the matching of the interior and exterior demands the continuity of the first and second fundamental forms on the bounding hyper surface.(Note that, if there are matters on the boundary then the second fundamental form has a jump discontinuity over the boundary):

The continuity of the first fundamental form gives

$$i)\; B\Lambda_\Sigma = R(t, r_\Sigma) = R_\Sigma \;\; \text{(say)} \;\; ii)\; e^\gamma dt = \left[A^2 - B^2 \left(\frac{d\Lambda_\Sigma}{dT}\right)^2\right]^{\frac{1}{2}} dT = d\tau \tag{54}$$

For matching of the second fundamental form we need extrinsic curvature which is given by

$$K_{\mu\nu}{}^{\pm} = -n_\sigma{}^{\pm}\; x^\sigma{}_{,\zeta^\mu,\zeta^\nu} - n_\sigma{}^{\pm}\; \Gamma_{\beta\delta}{}^{\sigma} x^\beta{}_{,\zeta^\mu} x^\gamma{}_{,\zeta^\nu} \tag{55}$$

where ζ^μ are the intrinsic coordinates on $\sum$, x^σ are the coordinates of $V^\pm$, the (,) denotes partial differentiation and the expression for the unit normal n_σ to the hypersurface (f =constant) is given by

$$n_\sigma = f_{,\sigma} / \left[g^{\mu\nu} f_{,\mu} f_{,\nu}\right]^{\frac{1}{2}} \tag{56}$$

In the present matching model the explicit form of the unit normal is given by

$$n_\sigma{}^- = \left(0, e^{\alpha(t,r_\Sigma)}, 0, 0\right) \quad \text{and} \quad n_\sigma{}^+ = \left(-\frac{d\Lambda_\Sigma}{dT}, 1, 0, 0\right) \frac{AB}{\sqrt{A^2 - B^2 \left(\frac{d\Lambda_\Sigma}{dT}\right)^2}} \tag{57}$$

Then the non-vanishing components of the extrinsic curvature in V^- and V^+ have the expressions

$$V^- : \; K_{\theta\theta}{}^- = Sin^{-2}\theta K_{\phi\phi}{}^- = (RR' e^{-\psi})_{\Sigma^-} \tag{58}$$

$$V^+ : \; K_{TT}{}^+ = \frac{AB}{\left(A^2 - B^2 \left(\frac{d\Lambda_\Sigma}{dT}\right)^2\right)^{\frac{3}{2}}} \left\{ \frac{B}{A^2} \frac{\partial B}{\partial T} \left(\frac{d\Lambda_\Sigma}{dT}\right)^3 + \left(\frac{2}{A}\frac{\partial A}{\partial \Lambda} - \frac{1}{B}\frac{\partial B}{\partial \Lambda}\right) \left(\frac{d\Lambda_\Sigma}{dT}\right)^2 \right.$$

$$\left. + \left(\frac{1}{A}\frac{\partial A}{\partial T} - \frac{2}{B}\frac{\partial B}{\partial T}\right)\left(\frac{d\Lambda_\Sigma}{dT}\right) - \frac{d^2\Lambda_\Sigma}{dT^2} - \frac{A}{B^2}\frac{\partial A}{\partial \Lambda} \right\}_{\Sigma^+} \tag{59}$$

and

$$K_{\theta\theta}{}^+ = \frac{AB^2\Lambda_\Sigma}{\left(A^2 - B^2 \left(\frac{d\Lambda_\Sigma}{dT}\right)^2\right)^{\frac{1}{2}}} \left[\frac{\Lambda_\Sigma}{A^2} \left(\frac{\partial B}{\partial T}\right) \frac{d\Lambda_\Sigma}{dT} + \frac{(B + \frac{\partial B}{\partial \Lambda}\Lambda_\Sigma)}{B^2} \right]_{\Sigma^+} \tag{60}$$

Hence matching of the second fundamental form demands

$$K_{TT}{}^+ = 0 \quad \text{and} \quad K_{\theta\theta}{}^+ = (RR' e^{-\psi})_{\Sigma^-} \tag{61}$$

In particular, if we take V^+ as the Schwarzschild metric, namely,

$$V^+ : \; ds_+{}^2 = -A(R) dT^2 + \frac{1}{A(R)} dR^2 + R^2 d\Omega^2 \tag{62}$$

with $A(R) = 1 - \frac{2M}{R}$

then it can be seen easily by a straight forward calculation that

$$K_{TT}{}^+ = 0 \quad \text{and} \quad K_{\theta\theta}{}^+ = \left(AR\frac{dT}{d\tau}\right)_{\Sigma^+} \tag{63}$$

So the continuity of $K_{\theta\theta}$ over Σ i.e, $K_{\theta\theta}{}^+ - K_{\theta\theta}{}^- \mid_\Sigma = 0$ simplifies to

$$2M = R_\Sigma \left[1 - e^{-2\psi} R'^2 + e^{-2\gamma} \dot{R}^2\right]_\Sigma \tag{64}$$

which is the definition of the mass function given in equation (11). Hence for a smooth matching of the interior (spherical star) and exterior space-times (Schwarzschild), the interior mass function at the surface must be equal to the Schwarzschild mass of the exterior vacuum.

To study the formation of the horizon in this model we have from equation (35)

$$R = m(r)$$

i.e, only one apparent horizon exists in this case.

In particular, if the exterior geometry is Schwarzschild, then apparent horizon is the Schwarzschild horizon. Further, as before, the time difference between the formation of apparent horizon at comoving coordinate r and the time of formation of singularity of a shell at comoving coordinate r is given by ($r << r_\Sigma$)

$$t_{ah}(r) - t_s(r) = -\int_0^{v_{ah}(r)} \frac{\sqrt{v}dv}{\sqrt{\left(1 - \frac{2p_{t2}}{\rho_0} r^2 v^2\right)\left(\frac{\rho_0}{3} + b_{02}v - \frac{2p_{t2}}{\rho_0} b(r) v^3\right)}} \tag{65}$$

Thus locally, if $t_{ah}(r) < t_s(r)$ then final state of collapse will be a black hole while if $t_{ah}(r) \geq t_s(r)$ then final singularity will be naked. The above time difference depends on the tangential pressure and total matter density. Thus dark energy in the form of anisotropic fluid influences the collapsing process and also the end state of collapse.

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